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Matching Conditions and Duality in $\mathbf{N = 1}$ SUSY Gauge Theories in the Conformal Window

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Abstract

We discuss duality in $N = 1$ SUSY gauge theories in Seiberg's conformal window, $(3N_c/2) < N_f < 3N_c$. The 't Hooft consistency conditions – the basic tool for establishing the infrared duality – are considered taking into account higher order α corrections. The conserved (anomaly free) R current is built to all orders in α. Although this current contains all orders in α the 't Hooft consistency conditions for this current are shown to be one-loop. This observation thus justifies Seiberg's matching procedure. We also briefly discuss the inequivalence of the "electric" and "magnetic" theories at short distances.

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1 Introduction

In this paper we discuss the infrared duality between different ("electric" and "magnetic") $N = 1$ SUSY gauge models observed in Ref. [1]. Supersymmetric (SUSY) gauge theories are unique examples of non-trivial four-dimensional theories where some dynamical aspects are exactly tractable. The first results of this type – calculation of the gluino condensate and the Gell-Mann-Low function – were obtained in the early eighties [2, 3]. The interest to the miraculous features of the supersymmetric theories was revived after the recent discovery [1], [4]-[6] of a rich spectrum of various dynamical scenarios that may be realized with a special choice of the matter sector (for a review see Ref. [7]). The basic tools in unraveling these scenarios are:

- instanton-generated superpotentials which may or may not lift degeneracies along classically flat directions [8];
- the NSVZ β functions [2, 3];
- the property of holomorphy in certain parameters [9, 10, 11];
- various general symmetry properties, i.e. the superconformal invariance at the infrared fixed points and its consequences [1].

A beautiful phenomenon revealed in this way is the existence of a generalized "electric-magnetic" duality in $N = 2$ [12] and some versions of $N = 1$ theories [1].

In Ref. [1] it was argued that $SU(N_c)$ and $SU(N_f - N_c)$ gauge theories with N_f flavors (and a specific Yukawa interaction in the "magnetic" theory) flow to one and the same limit in the infrared asymptotics. If $3N_c/2 < N_f < 3N_c$ both theories are conformal – this is the so called conformal window. In other words, for these values of N_f the Gell-Mann-Low functions of both theories vanish at critical values of the coupling constants. In particular, for the "electric" $SU(N_c)$ gauge theory with N_f flavors in the fundamental representation the β function corresponding to the gauge coupling has the following form [2, 3]:

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\,\frac{3N_c - N_f(1-\gamma(\alpha))}{1-(N_c\alpha/2\pi)}\,, \tag{1}$$

where $\gamma(\alpha)$ is the anomalous dimension of the matter field,

$$\gamma(\alpha) = -\frac{N_c^2-1}{2N_c}\frac{\alpha}{\pi} + O(\alpha^2)\,. \tag{2}$$

The critical value of the coupling constant α_* is determined by the zero of the β function,

$$\gamma(\alpha_*) = \gamma_* = 1 - 3\frac{N_c}{N_f}\,,\quad \alpha_* < \frac{2\pi}{N_c}\,. \tag{3}$$

According to Ref. [1] two dual theories, to be discussed below, have the following content: the first one ("electric") has $SU(N_c)$ gauge group and N_f massless flavors while its dual theory ("magnetic") has $SU(N_f - N_c)$ gauge group, the same number N_f of massless flavors (but with different $U(1)$ quantum numbers) plus N_f^2 colorless massless "meson" fields [1]. The electric and magnetic theories are supposed to have one and the same infrared limit (although their behavior at short distances is distinct; see below). Moreover, the electric theory is weakly coupled near the right edge of the window where the magnetic one is strongly coupled, and *vice versa*.

The main tool used in [1] for establishing the infrared equivalence is the 't Hooft consistency condition [13]. As was first noted in [14] the chiral anomaly implies the existence of the infrared singularities in the matrix elements of the axial current (and the energy-momentum tensor) which are fixed unambiguously. Therefore, even if we do not know how to calculate in the infrared regime the infrared limit of the theory should be arranged in such a way as to match these singularities.

The standard consideration is applicable only to the so called external anomalies. One considers the currents (corresponding to global symmetries) which are non-anomalous in the theory *per se*, but acquire anomalies in weak external backgrounds. For instance, in QCD with several flavors the singlet axial current is internally anomalous – its divergence is proportional to $G\tilde{G}$ where G is the gluon field strength tensor. The non-singlet currents are non-anomalous in QCD but become anomalous if one includes the photon field, external with respect to QCD. The anomaly in the singlet current does not lead to the statement of the infrared singularities in the current while the anomaly in the non-singlet currents does. Thus, for the 't Hooft matching one usually considers only the set of external anomalies.

At first sight in the conformal window the set of the external anomalies includes extra currents due to the vanishing of the β function at the conformal point, $\beta(\alpha_*) = 0$. In the framework of supersymmetry it means that the trace of the stress tensor θ^μ_μ vanishes as well as the divergence of some axial current entering the same supermultiplet as the stress tensor $\theta_{\mu\nu}$. Therefore, the idea that immediately comes to one's mind is that the standard matching conditions should be supplemented by the new singlet axial current. Actually, this was the starting point in the first version of this paper. The point is false, however.

Our analysis shows that:

(i) The number of the matching conditions at the conformal point is not expanded and is the same as in Ref. [1].

(ii) However, what changes is the form of the conserved R current, to be used in the matching conditions; coefficients in the definition of the conserved R current are α dependent, i.e. are affected by higher loops.

(iii) Although the R current is different from the naive one (where α is set equal to zero) *consequences* for the matching conditions and superpotentials remain intact provided one takes into account higher order corrections consistently everywhere, together with the specific form of the NSVZ β function. The crucial observation is the fact that the conserved R current which includes all orders in coupling constants

still yields the 't Hooft consistency conditions with no higher loop corrections. The fact that higher orders in α have no impact in some relations is due to the existence of a new type of holomorphy in the effective Lagrangian for the anomalous triangles in external fields.

The paper is organized as follows. In Sec. 2 we briefly review those results of Ref. [1] which are relevant for our analysis, introducing notations to be used throughout the paper. In Sec. 3 we discuss the construction of the conserved (anomaly free) R current to all orders in the coupling constants. Sec. 4 treats the anomaly matching conditions at the multi-loop level. It is shown here that the higher order corrections present in the R current are canceled in the 't Hooft consistency conditions for the external backgrounds. Sec. 5 explains the cancellation of the higher order corrections in the triangles for the R current to the baryon currents. In Sec. 6 we discuss the selection rules for the superpotentials. In Sec. 7 we comment on inequivalence of the electric and magnetic theories at short distances. Sec. 8 is devoted to incorporating the Yukawa couplings in the analysis of the infrared fixed points. The anomalous dimension of the M field is derived here from the requirement of the conformal symmetry in the infrared limit.

2 One-loop anomaly matching condition

The action of the electric theory is

$$S = \frac{1}{2g^2}\int d^4x d^2\theta \; \mathrm{Tr}\, W^2 + \frac{Z}{4}\sum_f \int d^4x d^4\theta \; \left(\bar{Q}_f^\dagger e^V \bar{Q}_f + Q_f^\dagger e^{-V} Q_f \right) \tag{4}$$

where Q_f and $\bar{Q}_f$ are the matter chiral superfields in the N_c and $\bar{N}_c$ color representations, respectively. The subscript f is the flavor index running from 1 to N_f. The theory has the following global symmetries free from the internal anomalies:

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \tag{5}$$

where the quantum numbers of the matter multiplets with respect to these symmetries are as follows

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_f	0	1	$(N_f - N_c)/N_f$
$\bar{Q}$	0	$\bar{N}_f$	-1	$(N_f - N_c)/N_f$

Table 1

The $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ transformations act only on the matter fields in an obvious way, and do not affect the superspace coordinate θ. As for the extra global symmetry $U(1)_R$ it is defined in such a way that it acts nontrivially on the

supercoordinate θ and, therefore, acts differently on the spinor and the scalar or vector components of superfields. The R charges in the Table 1 are given for the lowest component of the chiral superfields.

The notion of the R symmetries was originally introduced in Ref. [15]. The R current considered in Ref. [1] is a conserved current that is free from the triangle anomaly at the one-loop level. At the classical level there are two conserved axial currents. One of them – sometimes the corresponding symmetry is called R_0 – is the axial current entering the same supermultiplet as the energy-momentum tensor and the supercurrent. The R_0 current is classically conserved if all matter fields are massless; at the quantum level, generically, it acquires the internal anomaly proportional to $\beta(\alpha)G\tilde{G}$, see Refs. [17, 16, 9]. Another one is the flavor singlet current of the matter field. The anomaly of the latter current is purely one-loop. First it was shown in Ref. [17], later an independent analysis of this anomaly was carried out by Konishi *et al* [18, 19]. The corresponding expression is usually called the Konishi relation.

Seiberg's R charge refers to a combination of these two currents chosen in such a way as to ensure cancellation of the internal anomaly at one loop.(Let us parenthetically note the simplest example of conserved R current in the Abelian gauge theory was found long ago in Ref. [17].) This nonanomalous R symmetry transforms superfields in the following way:

$$W(\theta) \to e^{-i\epsilon} W(e^{i\epsilon}\theta)\,,$$

$$Q(\theta) \to e^{i\epsilon(N_c-N_f)/N_f} Q(e^{i\epsilon}\theta)\,, \quad \bar{Q}(\theta) \to e^{i\epsilon(N_c-N_f)/N_f} \bar{Q}(e^{i\epsilon}\theta)\,. \tag{6}$$

In the component form the R transformations are

$$\lambda \to e^{-i\epsilon}\lambda\,, \quad \psi(\bar{\psi}) \to e^{i\epsilon N_c/N_f}\psi(\bar{\psi}), \quad \phi(\bar{\phi}) \to e^{i\epsilon(N_c-N_f)/N_f}\phi(\bar{\phi}) \tag{7}$$

where $\phi(\bar{\phi})$ and $\psi(\bar{\psi})$ are the scalar and fermion components of chiral superfields Q and $\bar{Q}$.

The conserved R^S current is defined in [1] as

$$R^S_{\alpha\dot{\alpha}} = \frac{2}{g^2}\mathrm{Tr}\,(\lambda^\dagger_{\dot{\alpha}}\lambda_\alpha) - \frac{N_c}{N_f}\sum_f \psi^{f\dagger}_{\dot{\alpha}}\psi^f_\alpha - \frac{N_c}{N_f}\sum_f \bar{\psi}^{f\dagger}_{\dot{\alpha}}\bar{\psi}^f_\alpha \tag{8}$$

where α and $\dot{\alpha}$ are the standard spinor indices and $\mathrm{Tr}\,(\lambda^\dagger_{\dot{\alpha}}\lambda_\alpha) = \frac{1}{2}\lambda^{\dagger\,a}_{\dot{\alpha}}\lambda^a_\alpha$. Here and below the contribution of the scalars in the currents is consistently omitted.

It was suggested [1] that for $3N_c/2 < N_f < 3N_c$ there is another (magnetic) theory with the same number of flavors N_f but different color group, $SU(N_f-N_c)$, in which one has an additional "meson" supermultiplets M^i_j $(i,j = 1, ..., N_f)$. (Below, to distinguish the quark and gluon fields of the magnetic theory from those of the electric one the former will be marked by tilde.)

The quantum numbers of the new chiral quark superfields and the meson superfield M with respect to the global symmetries $SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)_R$ are as follows:

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\bar{N}_f$	0	$N_c/(N_f-N_c)$	N_c/N_f
$\bar{q}$	0	N_f	$-N_c/(N_f-N_c)$	N_c/N_f
M	N_f	$\bar{N}_f$	0	$2(N_f-N_c)/N_f$

Table 2

where the quantum numbers for meson field M are defined from the superpotential

$$\mathcal{W} = f\, M^i_j q_i \bar{q}^j\ , \tag{9}$$

and the conserved R current must be defined as

$$\tilde{R}^S_{\alpha\dot{\alpha}} = \frac{2}{g^2}\mathrm{Tr}\,(\tilde{\lambda}^\dagger_{\dot{\alpha}}\tilde{\lambda}_\alpha) - \frac{N_f-N_c}{N_f}\sum_f \tilde{\psi}^{f\dagger}_{\dot{\alpha}}\tilde{\psi}^f_\alpha - \frac{N_f-N_c}{N_f}\sum_f \tilde{\bar{\psi}}^{f\dagger}_{\dot{\alpha}}\tilde{\bar{\psi}}^f_\alpha$$

$$+\frac{N_f-2N_c}{N_f}\mathrm{Tr}\,(\chi^\dagger_{\dot{\alpha}}\chi_\alpha) \tag{10}$$

where $\tilde{\lambda}$ and $\tilde{\psi}$,$\tilde{\bar{\psi}}$ are dual gluino and quarks and χ are the fermions from the supermultiplet M.

Let us also note that if the number of the dual colors $N_f - N_c$ is the same as N_c, i.e. $N_f = 2N_c$, then the R charge of M is zero. At this point, $N_f = 2N_c$, the electric and magnetic theories look self-dual. As we will see shortly, the actual situation is more complicated. The fermions χ do not decouple from the R current defined beyond one loop.

The electric and magnetic theories described above are equivalent in their respective infrared (IR) conformal fixed points – with the choice of the quantum numbers above the highly non-trivial 't Hooft anomaly matching conditions for the currents corresponding to $SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)_R$ are satisfied. If the $SU(N_c)$ theory is weakly coupled at the conformal point α_* the dual $SU(N_f-N_c)$ theory will be strongly coupled, and it is only the anomaly relations which can be compared, because they can be reliably calculated in both theories. The presence of fermions from the meson multiplet M is absolutely crucial for this matching. Specifically, one finds for the one-loop anomalies in both theories [1]:

$$\begin{aligned}
SU(N_f)^3 &\;\rightarrow\; N_c d^{(3)}(N_f)\\
SU(N_f)^2U(1)_R &\;\rightarrow\; -\frac{N_c^2}{N_f}d^{(2)}(N_f)\\
SU(N_f)^2U(1)_B &\;\rightarrow\; N_c d^{(2)}(N_f)\\
U(1)_B^2U(1)_R &\;\rightarrow\; -2N_c^2\\
U(1)_R^3 &\;\rightarrow\; N_c^2-1-2\frac{N_c^4}{N_f^2}\\
U(1)_R &\;\rightarrow\; -N_c^2-1
\end{aligned} \tag{11}$$

where the constants $d^{(3)}$ and $d^{(2)}$ were introduced in [1] and are related to the traces of three and two $SU(N_f)$ generators.

For example, in the $U(1)_R^3$ anomaly in the electric theory the gluino contribution is proportional to $N_c^2 - 1$ and that of quarks to $-(N_c/N_f)^3 2N_f N_c = -2N_c^4/N_f^2$; altogether $N_c^2 - 1 - 2N_c^4/N_f^2$ as in (11). In the dual theory one gets from gluino and quarks $\tilde{q}$, $\bar{\tilde{q}}$ another contribution, $(N_f - N_c)^2 - 1 - 2(N_f - N_c)^4/N_f^2$. Then the fermions χ from the meson multiplet M (see Eq. (10)) add extra $[(N_f - 2N_c)/N_f]^3 N_f^2$, which is precisely the difference.

The last line in Eq. (11) corresponds to the anomaly of the R current in the background gravitational field,

$$\partial^{\alpha\dot{\alpha}} J^R_{\alpha\dot{\alpha}} \sim \epsilon_{\mu\nu\lambda\delta} R^{\mu\nu\sigma\rho} R^{\lambda\delta}_{\sigma\rho} \,. \tag{12}$$

In the electric $SU(N_c)$ theory the corresponding coefficient is

$$N_c^2 - 1 - 2(N_c/N_f)N_c N_f = -N_c^2 - 1$$

while in the magnetic theory it is $-(N_f - N_c)^2 - 1$ from quarks and gluinos and $[(N_f - 2N_c)/N_f]N_f^2$ from the M fermions, i.e. in the sum again $-N_c^2 - 1$.

As was discussed above, in accordance with the standard logic, the set of the matching conditions above includes only those currents that do not have internal anomalies. The number of the matching conditions is rather large and the fact they are satisfied with the given field content is highly non-trivial.

3 Conserved R currents

In the consideration above it was crucial that there exists a singlet axial current (R current) whose conservation is preserved at the quantum level. The particular form of the current (8) assumes that the coefficients are α independent numbers. We will show below that this is not the case if higher loops are included and we will determine the coefficients in terms of the anomalous dimensions γ of the matter fields. This definition is consistent with the fact that the anomaly in the divergence of the R^0 current is multi-loop.

Let us consider first the electric theory. At the classical level there exist two conserved singlet currents. The first one is the member of the supermultiplet containing the stress tensor and the supercurrent [20], it has the following universal form:

$$R^0_{\alpha\dot{\alpha}} = \frac{2}{g^2}\mathrm{Tr}\,(\lambda^\dagger_{\dot{\alpha}}\lambda_\alpha) - \frac{1}{3}\left(\sum_f \psi^{f\dagger}_{\dot{\alpha}}\psi^f_\alpha + \sum_f \bar{\psi}^{f\dagger}_{\dot{\alpha}}\bar{\psi}^f_\alpha\right) \,. \tag{13}$$

The current $R^0_{\alpha\dot{\alpha}}$ is the lowest component of the superfield $J^0_{\alpha\dot{\alpha}}$ (see Ref. [16]),

$$J^0_{\alpha\dot{\alpha}} = -\frac{2}{g^2}\mathrm{Tr}\,\left(W_\alpha e^V W^\dagger_{\dot{\alpha}} e^{-V}\right) + \frac{Z}{12}\Big\{\Big[(D_\alpha(e^{-V}Q))e^V \bar{D}_{\dot{\alpha}}(e^{-V}Q^\dagger)$$

$$+Qe^{-V}D_\alpha(e^V\bar{D}_{\dot\alpha}(e^{-V}Q^\dagger)) + Q\bar{D}_{\dot\alpha}(e^{-V}D_\alpha Q^\dagger) - (Q\to Q^\dagger, V\to -V)\Big]$$

$$+(Q\to\bar{Q}\,,\ V\to -V)\Big\}\,. \tag{14}$$

The current (13) corresponds to the transformation of the superfields

$$W(\theta)\to e^{3i\alpha}W(e^{-3i\alpha}\theta)\,,$$

$$Q\,(\theta)\to e^{2i\alpha}Q\left(e^{-3i\alpha}\theta\right)\,,\ \ \bar{Q}\,(\theta)\to e^{2i\alpha}\bar{Q}\left(e^{-3i\alpha}\theta\right)\,. \tag{15}$$

In components this means

$$\lambda\to e^{3i\alpha}\lambda,\quad \psi(\bar\psi)\to e^{-i\alpha}\psi(\bar\psi),\quad \phi(\bar\phi)\to e^{2i\alpha}\phi(\bar\phi)\,; \tag{16}$$

this symmetry exists in the presence of the Yukawa couplings of the form $c_{ijk}Q_iQ_jQ_k$.

The second classically conserved current, $K_{\alpha\dot\alpha}$, which we will refer to as the Konishi current, is built from the matter fields only:

$$K_{\alpha\dot\alpha} = \frac{1}{3}\sum_f \psi^{f\dagger}_{\dot\alpha}\psi^f_\alpha + \frac{1}{3}\sum_f \bar\psi^{f\dagger}_{\dot\alpha}\bar\psi^f_\alpha\,. \tag{17}$$

Note, that although the Konishi current (17) superficially looks identical to the second term in equation (13), actually they are different – the (omitted) contributions of the scalars in (13) and (17) are different. The current (17) is the lowest component of the superfield

$$-\frac{Z}{12}\Big\{\Big[(D_\alpha(e^{-V}Q))e^V\bar{D}_{\dot\alpha}(e^{-V}Q^\dagger) - \frac{1}{2}Qe^{-V}D_\alpha(e^V\bar{D}_{\dot\alpha}(e^{-V}Q^\dagger)) - \frac{1}{2}Q\bar{D}_{\dot\alpha}(e^{-V}D_\alpha Q^\dagger)$$

$$-(Q\to Q^\dagger, V\to -V)\Big] + (Q\to\bar{Q}\,,\ V\to -V)\Big\}\,. \tag{18}$$

The Konishi current corresponds to the transformation of the superfields:

$$W(\theta)\to W(\theta),\quad Q\,(\theta)\to e^{i\beta}Q\,(\theta)\,,\quad \bar{Q}\,(\theta)\to e^{i\beta}\bar{Q}\,(\theta) \tag{19}$$

Both currents R^0_μ and K_μ have anomalies at the quantum level. The anomaly in the R^0 current is multi-loop and in the K current is one-loop. In the operator form the anomalies can be written as follows [9]:

$$\bar{D}^{\dot\alpha}J^0_{\alpha\dot\alpha} = -\frac{1}{24}D_\alpha\left[\frac{3N_c-N_f}{2\pi^2}\mathrm{Tr}\,W^2 + \gamma Z\bar{D}^2\sum_f\left(Q^\dagger_f e^V Q_f + \bar{Q}^\dagger_f e^{-V}\bar{Q}_f\right)\right] \tag{20}$$

and

$$\bar{D}^2 Z\sum_f\left(Q^\dagger_f e^V Q_f + \bar{Q}^\dagger_f e^{-V}\bar{Q}_f\right) = \frac{N_f}{2\pi^2}\mathrm{Tr}\,W^2 \tag{21}$$

Here the anomalous dimension γ is defined as

$$\gamma = -\frac{d\,\ln Z}{d\,\ln \mu}\,,$$

and in one loop is given by equation (2). Equation (21) is the Konishi anomaly [18]-[19]. The second term in the right-hand side of equation (20) is due to higher-loop effects and represents the violation of the holomorphy of the effective Lagrangian.

By virtue of the Konishi anomaly (21) the second term in the right-hand side of equation (20) is transformed into the same gauge anomaly. The corresponding divergence in the R^0 current looks as follows:

$$\partial^\mu R^0_\mu = \frac{1}{48\pi^2}\,\left[3N_c - N_f(1-\gamma)\right]\,G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \tag{22}$$

where the coefficient in the square bracket is the numerator of the NSVZ β-function (1). The denominator will appear after taking the matrix element of the operator $G\tilde{G}$.

Let us write down in parallel the anomaly in the matter current

$$\partial^\mu K_\mu = \frac{1}{48\pi^2}\,N_f\,G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \tag{23}$$

From these anomalies we easily recover the form of the *only* conserved R current in the theory

$$R_\mu = R^0_\mu + \left[1 - \frac{3N_c}{N_f} - \gamma\right] K_\mu\,. \tag{24}$$

One could give an alternative derivation of the very same conserved R current considering the mixing matrix we will between R^0 and K currents which arise already at the one-loop level. Diagonalizing the mixing we can find two renormalization group (RG) invariant currents, one of which coincides with the conserved R_μ and the second one, which is not conserved, is $(1-\gamma)K_\mu$ (see Ref. [9] for details).

Consider now how the conserved current R_μ looks like in two limits. In the conformal point, when $\alpha = \alpha_*$ the coefficient in front of K_μ vanishes (see Eq. (3)). Thus in the infrared limit the R current flows to R^0. On the other hand in the extreme ultraviolet (UV) limit $\alpha(\mu) \to 0$ (i.e. $\gamma(\alpha) \to 0$) the R current flows to the Seiberg R^S current (8). Therefore the genuine R current interpolates between the R^S and R_0 currents.

Keeping in mind that in the magnetic theory there are two distinct matter fields with a superpotential, let us generalize the procedure of construction of the conserved R current to the case with some number of matter superfields S_i and a nonvanishing superpotential $\mathcal{W}$.

The definition of the R^0 current is general since it has a geometrical nature,

$$\tilde{R}^0_{\alpha\dot{\alpha}} = \frac{2}{g^2}\mathrm{Tr}\,(\tilde{\lambda}^\dagger_{\dot{\alpha}}\tilde{\lambda}_\alpha) - \frac{1}{3}\sum_i \psi^{i\dagger}_{\dot{\alpha}}\psi^i_\alpha\,, \tag{25}$$

where ψ^i is the fermionic component of the chiral superfield S^i. In the presence of the superpotential $\mathcal{W}$ there are two sources of the current nonconservation: the first source is possible classical nonconservation due to $\mathcal{W} \neq 0$, the second one is the quantum anomaly. In the superfield notations we have the following generalization of the equation (20):

$$\bar{D}^{\dot{\alpha}} J^0_{\alpha\dot{\alpha}} = \frac{1}{3} D_\alpha \left\{ \left[3\mathcal{W} - \sum_i S_i \frac{\partial \mathcal{W}}{\partial S_i} \right] - \left[\frac{b}{16\pi^2} \mathrm{Tr}\, W^2 + \frac{1}{8} \sum_i \gamma_i Z_i \bar{D}^2 \left(S_i^\dagger e^V S_i \right) \right] \right\} \tag{26}$$

where $b = 3N_c - \sum_i T_i$ is the first coefficient of the β function and invariants T_i characterize the gauge group representation of the field S_i (they are defined as $\mathrm{Tr}\, t^a t^b = T_i \delta^{ab}$ where t^a are the matrices of the group generators).

We can also construct the Konishi current

$$K^i_{\alpha\dot{\alpha}} = \frac{1}{3} \psi^{i\dagger}_{\dot{\alpha}} \psi^i_\alpha \, . \tag{27}$$

for each superfield S_i. The divergence of this current is given by the generalized Konishi relation:

$$\frac{1}{8} \bar{D}^2 \left(Z_i S_i^\dagger e^V S_i \right) = \frac{1}{2} S_i \frac{\partial \mathcal{W}}{\partial S_i} + \frac{T_i}{16\pi^2} \mathrm{Tr}\, W^2 \, . \tag{28}$$

(A comment on the literature: the anomaly in the current (26) was expressed in terms of anomalous dimensions γ_i in Ref. [16, 9]; for a recent instructive discussion which includes the classical nonconservation [20], see Ref. [21].)

Let us look for the conserved R current as a linear combination of the currents $R^0_{\alpha\dot{\alpha}}$ and $K^i_{\alpha\dot{\alpha}}$:

$$R_{\alpha\dot{\alpha}} = R^0_{\alpha\dot{\alpha}} + c_i K^i_{\alpha\dot{\alpha}} \, . \tag{29}$$

The divergence of this current can be immediately found from Eqs. (26) and (28),

$$\partial_\mu R^\mu = -\frac{4}{3} \left\{ \left[3\mathcal{W} - \sum_i S_i \frac{\partial \mathcal{W}}{\partial S_i} \left(1 + \frac{c_i + \gamma_i}{2} \right) \right] \Big|_G - \frac{\mathrm{Tr}\, W^2|_G}{16\pi^2} \left[b + \sum_i T_i \left(c_i + \gamma_i \right) \right] \right\} \, . \tag{30}$$

where the subscript G marks the G component of the chiral superfield, in particular, $\mathrm{Tr}\, W^2|_G = G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}/4$. All terms proportional to γ_i occur by virtue of the substitution of the Konishi relation (28) into (26).

For the conserved R current to exist both terms in Eq. (30), the superpotential term and the one with W^2, must vanish. All higher order corrections reside in the anomalous dimensions γ_i which depend on the gauge coupling constant α and the constants in the superpotential $\mathcal{W}$.

Let us first omit these higher order corrections, i.e. put $\gamma_i = 0$. For a generic superpotential there may be no conserved currents at all. This situation is of no interest to us, so we assume that one (or more) conserved currents exist. Let us denote by $c_i^{(0)}$ the set of the coefficients c_i ensuring the vanishing of ∂R at $\gamma_i = 0$.

(These coefficients $c_i^{(0)}$ are rational numbers which in many cases were found by Seiberg *et al.*)

Now let us switch on the higher order corrections, $\gamma_i \neq 0$. It is crucial that the anomalous dimensions γ_i appear only in the combination $c_i + \gamma_i$. This means that the coefficients c_i ensuring the current conservation at $\gamma_i \neq 0$ are different from $c_i^{(0)}$ only by a shift by $(-\gamma_i)$,

$$c_i = c_i^{(0)} - \gamma_i \tag{31}$$

Equation (24) given above is a particular example of this general result with $c^{(0)} = 1 - (3N_c/N_f)$. As another illustration let us consider the magnetic theory. In this case we have two Konishi currents,

$$\tilde{K}^q_{\alpha\dot{\alpha}} = \frac{1}{3}\sum_f \tilde{\psi}^{f\dagger}_{\dot{\alpha}}\tilde{\psi}^f_\alpha + \frac{1}{3}\sum_f \tilde{\tilde{\psi}}^{f\dagger}_{\dot{\alpha}}\tilde{\tilde{\psi}}^f_\alpha \ , \qquad \tilde{K}^M_{\alpha\dot{\alpha}} = \frac{1}{3}\mathrm{Tr}\,(\chi^\dagger_{\dot{\alpha}}\chi_\alpha) \tag{32}$$

where the notations have been introduced in Sec. 2. Equation (10) gives the values of $c_i^{(0)}$ in this case:

$$c_q^{(0)} = -\frac{1}{2}c_M^{(0)} = \frac{3N_c - 2N_f}{N_f} \ . \tag{33}$$

Higher order corrections will change these coefficients to

$$c_q = \frac{3N_c - 2N_f}{N_f} - \gamma_q \ ; \ \ c_M = -2\frac{3N_c - 2N_f}{N_f} - \gamma_M \ . \tag{34}$$

where γ_q and γ_M are the anomalous dimensions of the fields q and M respectively. Thus, the extra terms in the $\tilde{R}$ current as compared to R^S one (see Eq. (10)) are

$$\tilde{R}_{\alpha\dot{\alpha}} - R^S_{\alpha\dot{\alpha}} = -\gamma_q\tilde{K}^q_{\alpha\dot{\alpha}} - \gamma_M\tilde{K}^M_{\alpha\dot{\alpha}} \ . \tag{35}$$

One more comment concluding this section. The conserved R current is in the same supermultiplet with the stress-energy tensor $\theta_{\mu\nu}$ *only* in the infrared limit. Thus the relation $D = (3/2)R$ between the dimension D and the chiral R charge of the chiral superfield is valid only for the IR fixed point, but not for the UV one.

4 Cancellation of higher-loop corrections for the external anomalies

Now, when the conserved R current is constructed we can proceed to discussing the anomalies of this current in the background of weak external fields. As was mentioned in Sec. 1, these anomalies, via the 't Hooft consistency conditions, constraint the infrared behavior of the theory, and, thus, crucial in establishing the electric-magnetic duality. In Ref. [1] the anomaly relations were analyzed at the one-loop level. Since the duality can take place only in the conformal points where the coupling constants are not small a consideration of higher-loop corrections is absolutely

crucial. As we see, for instance, from Eq. (26), generally speaking, higher-loop corrections are present. Below we will demonstrate that in the external anomalies for the R current constructed above all higher order contributions cancel out.

To warm up we begin our consideration of the multi-loop effects starting from the example of $U(1)_R U(1)_B^2$ triangle. The definition of the baryon charges for the electric theory is given in section 2. We have α corrections both in the definition of the current R (see Eq. (24)) and in the anomalous triangle.

If we introduce an external field B_μ coupled to the baryon current then the anomaly for the R^0 current in the electric theory takes the form similar to equation (22),

$$\partial^\mu R^0_\mu = -\frac{1}{24\pi^2} N_f \, N_c \, (1-\gamma) \, B_{\mu\nu} \tilde{B}_{\mu\nu} \tag{36}$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Let us emphasize that α corrections to the anomalous triangle do not vanish and enter through the anomalous dimension $\gamma(\alpha)$ which is, in turn, related to the second term in the square brackets in Eq. (20). As far as the matter current K_μ is concerned here α corrections to the bare triangle vanish,

$$\partial^\mu K_\mu = \frac{1}{24\pi^2} N_f \, N_c \, B_{\mu\nu} \tilde{B}_{\mu\nu} \tag{37}$$

Assembling these two pieces together we find

$$\partial^\mu R_\mu = \frac{1}{16\pi^2} (-2N_c^2) \, B_{\mu\nu} \tilde{B}_{\mu\nu} \; . \tag{38}$$

One can see that the external anomaly in the R current has no α corrections – those coming from the definition of the current are canceled by the corrections in the triangle containing R^0_μ. Effectively the naive one-loop anomaly is preserved in the full multi-loop calculation in the case at hand.

As a matter of fact it is not difficult to generalize the assertion above to the case of arbitrary number of matter fields with a superpotential (e.g. in the magnetic theory we have two distinct sets of the matter fields – the magnetic quarks in the fundamental representation of the gauge group and color singlets).

Since the background field B_μ can be treated as an additional gauge field what is to be done, Eq. (30) must be supplemented by the extra term proportional to $W_B^2|_G = B_{\mu\nu} \tilde{B}^{\mu\nu}$. Namely,

$$\partial^\mu R_\mu = \frac{B_{\mu\nu} \tilde{B}^{\mu\nu}}{32\pi^2} \sum_i T_i^B (-1 + c_i + \gamma_i) \tag{39}$$

where T_i^B is equal to the trace of the square of the generators of the baryon charge, i.e. $T_i^B = B_i^2$ (B_i is the baryon charge of the field S_i). The analogy with the second term in Eq. (30) is quite transparent. The color generators are substituted by those of the baryon charge, and the coefficient b in the square brackets is substituted by $(-\sum T_i^B)$.

The coefficients c_i in Eq. (30) are chosen in such a way that the right-hand side of Eq. (30) vanishes. This vanishing is achieved provided $c_i+\gamma_i = c_i^{(0)}$, see Eq. (31). It is important that the very same combination, $c_i+\gamma_i$, enters Eq. (39). This means that the external baryonic anomaly reduces to

$$\partial^\mu R_\mu = \frac{B_{\mu\nu}\tilde{B}^{\mu\nu}}{32\pi^2}\sum_i T_i^B(-1+c_i^{(0)}) \tag{40}$$

Thus, the higher order corrections obviously drop out.

If the gravitational field is considered as external the calculation of the corresponding triangle is very similar to that in the external B_μ field. The only difference compared to the case of the baryon current is the substitution of T_i^B by unity,

$$\partial^\mu(\sqrt{-g}R_\mu) \;=\; \frac{1}{192\pi^2}\left[(N_c^2-1)-\frac{1}{3}\sum_i(-1+c_i^{(0)})\right]\epsilon_{\mu\nu\lambda\delta}R^{\mu\nu\sigma\rho}R^{\lambda\delta}_{\sigma\rho}\,, \tag{41}$$

In the electric theory $c_i^{(0)} = 1-(3N_c/N_f)$ while in the magnetic theory N_c is substituted by N_f-N_c and $c_{q,M}^{(0)}$ are given in Eq. (33).

The arguments can be of course repeated for the $U(1)_R\,SU(N_f)^2$ anomaly. What remains to be discussed is the anomalous $U(1)_R^3$ triangle. This case is harder to consider along the lines presented here. However, in the next section we will give some arguments of a more general nature indicating that this anomaly is also one-loop.

5 Cancellation of higher orders and holomorphy in the external field

Now we will show that the result derived above – cancellation of the higher order corrections in the external anomalies – has a very transparent interpretation in the language of the effective action in the external field. As an example of the external field one can keep in mind the field B_μ interacting with the baryonic charge, gravitational field and so on. The Wilson effective action is as follows:

$$\begin{aligned} S_W(\mu) = \frac{1}{16\pi^2}\left[\frac{8\pi^2}{g_0^2}-\left(3T_G\ln\frac{M_0}{\mu}-\sum_i T_i\ln\frac{M_i}{\mu}\right)\right]\int d^4xd^2\theta\ \mathrm{Tr}\,W^2+ \\ \frac{1}{16\pi^2}\left[\sum_i T_i^{(ext)}\ln\frac{M_i}{\mu}\right]\int d^4xd^2\theta\ W_{ext}^2+ \\ \sum_i\frac{Z_i}{4}\int d^4xd^4\theta\ S_i^\dagger e^V S_i+\left\{\int d^4xd^2\theta\ \mathcal{W}(S)+h.c.\right\} \end{aligned} \tag{42}$$

where $1/g_0^2$ is the inverse bare coupling constant and μ is the normalization point, the coefficients T_i are defined through the generators of the gauge group in the

representation i, $\mathrm{Tr}(t^a t^b) = T_i \delta^{ab}$, ($T_G$ is the invariant T_i in the adjoint representation). The masses M_0 and M_i are the regulator masses (one can keep in mind the supersymmetric Pauli-Villars regularization). In the covariant computation M_0 is the mass of the (chiral) ghost regulators, M_i is the mass of the S_i field regulator. Usually, it is assumed that all regulator masses are the same, $M_0 = M_i$. We keep them different for the purposes which will become clear shortly. Finally, W_{ext} is a superfield generalizing the stress tensor of the external gauge field in the same way as W generalizes $G_{\mu\nu}$ (in the previous section where the external baryonic current was considered as an example we dealt with W_B), the coefficients T_i^{ext} are defined similar to T_i for the generators corresponding to the interaction with the external field. The superpotential $\mathcal{W}(S_i)$ may or may not be present in each particular model.

The property of holomorphy in the Wilson effective action means that the coefficients in front of W^2 and W_{ext}^2 are given by one loop; higher order corrections in the coupling constants are absent. Higher orders enter only the Z factors; in taking the background field matrix elements of the last term in Eq. (42) higher orders in Z penetrate the answer.

Taking matrix elements of the operator action S_W we proceed to the c-number functional Γ, the generator of 1PI vertices. Let us first discuss what happens at one-loop level. Then the last two terms in Eq. (42) are irrelevant for the issue of anomalies under discussion, and the one-loop description is given by

$$\Gamma^{one-loop} = \frac{1}{16\pi^2}\left[\frac{8\pi^2}{g_0^2} - \left(3T_G \ln\frac{M_0}{\mu} - \sum_i T_i \ln\frac{M_i}{\mu}\right)\right]\int d^4x d^2\theta\ \mathrm{Tr}\, W^2 +$$
$$\frac{1}{16\pi^2}\left[\sum_i T_i^{(ext)} \ln\frac{M_i}{\mu}\right]\int d^4x d^2\theta\ W_{ext}^2\ . \tag{43}$$

From this expression one can easily read off anomalies by varying the regulator masses. For instance, the anomaly of R_0 current is obtained by applying the operator

$$M_0\frac{\partial}{\partial M_0} + \sum_i M_i \frac{\partial}{\partial M_i} \tag{44}$$

to the right-hand side of Eq. (43). The anomaly in the Konishi current K_i (see Eq. (27)) is generated by $M_i(\partial/\partial M_i)\Gamma^{one-loop}$.

Moreover, the same expression (43) demonstrates the existence of the conserved current R. Indeed, the first term in $\Gamma^{one-loop}$ is invariant under the action of the operator

$$M_0\frac{\partial}{\partial M_0} + \sum_i M_i \frac{\partial}{\partial M_i}(1 + c_i^{(0)}) \tag{45}$$

where coefficients $c_i^{(0)}$ were defined and discussed in Sect. 3. The non-invariance of the second term in $\Gamma^{one-loop}$ under the action of the operator (45) gives the external anomaly of the R current.

Now what happens if we proceed to higher loops? The occurrence of the Z factors in S_W manifests itself in Γ in the following way [9]:

$$\Gamma^{multi-loop} = \frac{1}{16\pi^2}\left[\frac{8\pi^2}{g_0^2} - \left(3T_G \ln\frac{M_0}{(g_0/g)^{\frac{2}{3}}\,\mu} - \sum_i T_i \ln\frac{M_i}{Z_i\,\mu}\right)\right]\int d^4x d^2\theta \ \mathrm{Tr}\, W^2 +$$

$$\frac{1}{16\pi^2}\left[\sum_I T_i^{(ext)} \ln\frac{M_i}{Z_i\mu}\right]\int d^4x d^2\theta \ W_{ext}^2 \ . \tag{46}$$

We write down here only the part of Γ containing W^2 and W^2_{ext}; the part with classical superpotential is omitted. The inclusion of higher orders resulted in substituting the regulator mass M_i by M_i/Z_i. The role of Z_i for the ghost regulator mass M_0 is played by $(g_0/g)^{\frac{2}{3}}$ (see Ref. [9]).

Invariance of the W^2 part of Γ still persists. However, the transformation under which it is invariant corresponds now to the action of the operator

$$\mathcal{M}_0\frac{\partial}{\partial\mathcal{M}_0} + \sum_i \mathcal{M}_i\frac{\partial}{\partial\mathcal{M}_i}(1 + c_i^{(0)}) \tag{47}$$

where

$$\mathcal{M}_i = \frac{M_i}{Z_i}\ , \quad \mathcal{M}_0 = \frac{M_0}{(g_0/g)^{\frac{2}{3}}}\ . \tag{48}$$

The application of operator (48) to the W^2_{ext} part of Γ yields the external anomaly. Since in this part M_i is also replaced by $\mathcal{M}_i = M_i/Z_i$ it is clear that the external anomaly remains one-loop.

A direct correspondence between the discussion of the external anomalies in Sect. 4 and the one given in this section is quite clear. However, the arguments of this section help demonstrate the general nature of the phenomenon. In particular it seems possible to deduce that the anomaly of the type $U(1)^3_R$ is also one-loop.

6 Superpotentials

We have demonstrated that the conserved R_μ current contains higher order terms. The question arises about the selection rules for different terms in superpotentials which were obtained without these complications [8].

The change of the form of the R current does not mean that these selection rules were incorrect. The same phenomenon of the cancellation of the higher order corrections as was described above takes place in the transformation laws of the chiral fields.

To elucidate this assertion it is instructive to analyze the form factor diagrams where the chiral matter scatters off the current R. Taken at the vanishing momentum transfer these graphs yield the R charge of the matter field. At a naive level one would start from the naive current $R\,|_{\gamma=0}$, with all γ terms discarded, draw the tree

graph plus two one-loop graphs (the diagram with the vertex correction and the diagram with the correction to the external line), and then one would conclude that the two one-loop graphs cancel each other in the same way similar diagrams cancel each other in the electric current in QED. This naive conclusion would be wrong! If the calculation is done supersymmetrically, in terms of the supergraphs, getting a non-vanishing result for the sum of the two one-loop graphs mentioned above is inevitable. The residual sum of these graphs is canceled, however, when one adds the γ term of the R current as the vertex insertion in the tree graph. Effectively this means that the R charge of the chiral matter is determined by the tree graph with $R\,|_{\gamma=0}$ at the vertex, i.e. coincides with Seiberg's answer. In other words the very same assertion can be phrased as follows: the commutator of the R charge with the (bare) matter fields S_i contains an anomalous part that cancels the γ terms in the definition of the conserved R charge. In particular, the commutator with the modular fields

$$\left[R,\ Q^i_f\bar{Q}_{i\,f'}\right]$$

remains the same as in the naive analysis with all γ's set equal to zero.

A nice illustration of how this cancellation works is provided by the simplest supersymmetric model, the massless Wess-Zumino model. This example was thoroughly analyzed in Ref. [16] and we summarize here the basic points. The action of the model is

$$S_{WZ} = \frac{Z}{4}\int d^4x d^4\theta\ {S_0}^{\dagger}S_0 + \left\{\int d^4x d^2\theta\ f_0 S_0^3 + h.c.\right\}\,, \tag{49}$$

where S_0 is the bare field and f_0 is the bare coupling constant. The R^0 current of this model is merely

$$-\frac{1}{3}\psi_{\dot\alpha}\psi_{\alpha}\,,$$

i.e. the lowest component of the superfield

$$J^0_{\alpha\dot\alpha} = \frac{Z}{12}\left\{\left[(D_\alpha S)\bar{D}_{\dot\alpha}(S^\dagger) + SD_\alpha\bar{D}_{\dot\alpha}S^\dagger + S\bar{D}_{\dot\alpha}D_\alpha S^\dagger - (S\to S^\dagger)\right]\right\}\,.$$

The R^0 current is classically conserved. The corresponding field transformation

$$S(\theta)\to e^{2i\alpha}S(e^{-3i\alpha}\theta)$$

leaves invariant both the kinetic and the superpotential terms in the Wess-Zumino action. However, since R^0 enters the same supermultiplet as the energy-momentum tensor it conservation is ruined at the quantum level when loops are included. The non-conservation is associated only with the occurrence of the Z factor in the kinetic term and can be interpreted as the anomalous non-invariance of the kinetic term under the phase rotation generated by the R^0 current. The superpotential term, written through the bare fields remains invariant – this follows from the famous non-renormalization theorem [22] which tells us that there are no quantum corrections to

the superpotential term and, hence, no dependence of this term on the normalization point μ. The anomalous divergence has the form

$$\partial_\mu R^0_\mu = \gamma \partial_\mu K_\mu$$

where K_μ is the Konishi current of the Wess-Zumino model,

$$K_\mu = \frac{1}{3}\psi_{\dot\alpha}\psi_\alpha\,,$$

or the lowest component of the superfield

$$-\frac{Z}{12}\left\{\left[(D_\alpha S)\bar{D}_{\dot\alpha}S^\dagger - \frac{1}{2}SD_\alpha\bar{D}_{\dot\alpha}S^\dagger - \frac{1}{2}S\bar{D}_{\dot\alpha}D_\alpha S^\dagger) - (S\to S^\dagger)\right]\right\}.$$

Up to a sign, the fermion parts of both currents look identical. Transferring $\gamma\partial_\mu K_\mu$ to the left-hand side of the anomaly relation we get the conserved R current. It is clear that the role of the term $-\gamma K_\mu$ in R_μ is to kill the anomalous non-invariance of the kinetic term under the phase rotation. It has no effect whatsoever on the transformation properties of the superpotential term, which is absolutely clear from the derivation of the anomaly. If we forget altogether about this anomalous non-invariance of the kinetic term (as one would do naively) and use the naive charge (in this case this is just the R^0 charge) we arrive at the correct conclusion concerning the invariance of the superpotential term. This situation is quite general.

The important lesson stemming from the analysis is as follows: the selection rules obtained for the superpotential terms from the naive (one-loop) R^S current are valid only provided we work in terms of the bare (unrenormalized) fields, so that all Z factors reside in the kinetic terms.

7 Comments on short distance behavior in the electric and magnetic theories

In this section we comment further on the question what is the precise meaning of the duality between the electric and magnetic theories. According to Ref. [1] it should be understood as equivalence of the infrared limits of both theories – i.e. large distance scaling of all correlation functions with the same anomalous dimensions for equivalent operators. Sometimes a stronger conjecture is made in the literature – the full equivalence of the electric and magnetic theories, at all distances. Here we will consider corresponding correlation functions at short distances in the electric and magnetic theories and explicitly demonstrate that they are different.

First of all, we must choose operators that correspond to each other in the electric and magnetic theories. No general relation between respective operators is known. The correspondence between operators in both theories may be complicated, even nonlocal. We can consider, however, the Noether currents which in both theories

correspond to one and the same symmetry. The simplest choice is the baryon number current.

Due to asymptotic freedom we can compare the corresponding correlation functions at short distances. The leading asymptotics is that of the corresponding free theories, and the disbalance is obvious. The only subtle point which deserves discussion is the presence of the additional Yukawa interaction in the magnetic theory.

The standard argument is as follows. The Yukawa coupling is not asymptotically free; therefore, at short distances it explodes, and this explosion precludes one from calculating the short-distance behavior in the magnetic theory.

In the next section we will explicitly demonstrate that the Yukawa coupling *is* asymptotically free in the conformal window, due to the contribution to the coupling renormalization coming from the exchange of the gauge fields, at least in some domain of the parameter space. Moreover, the magnetic theory has a nontrivial infrared fixed point for *both* couplings – the gauge and the Yukawa. (Otherwise, the corresponding theory would not be conformal at all, no scaling would be achieved at large distances, and no conformal window would exist.)

Thus we will compare two asymptotically free theories at short distances. In the asymptotic regime when all couplings become arbitrarily small a straightforward calculation yields the two-point correlation functions for the baryon currents at short distances,

$$\langle 0|\{J_B^\mu(x) J_B^\mu(0)\}|0\rangle = C\frac{2}{\pi^2}\,\frac{1}{x^6} + ..., \qquad x \to 0 \tag{50}$$

where the coefficient C depends on the theory. In the electric theory the baryon charge is 1 for Q and -1 for $\bar{Q}$, so the constant C can be easily shown to be

$$C_E = N_f N_c\,.$$

In the magnetic theory the quark baryon charge is $N_c/(N_f - N_c)$ and the constant C_M now is equal to

$$C_M = \left(\frac{N_c}{N_f - N_c}\right)^2 N_f(N_f - N_c) = \frac{N_c}{N_f - N_c} C_E\,.$$

Only for $N_f = 2N_c$ these constants are the same (which is evident because in this case the quark charges in the electric theory are the same as in the magnetic one). However, even at $N_f = 2N_c$, when the baryon two-point functions match, it is not difficult to show that the short distance behavior of some other two-point functions (e.g. induced by the $SU(N_f)$ currents) does not match. Even if we are not in the domain of the parameter space where the coupling f is asymptotically free, still f approaches small values at intermediate scales (if the gauge coupling is asymptotically free which is necessary for the consistency of the whole approach). This means that the disbalance between the corresponding correlation functions in the electric and magnetic theories can be established at this intermediate scale.

Thus one can see that this is the general case – the short distance behavior of correlation functions is different in both theories.

8 Conformal fixed points and the Yukawa couplings

Let us discuss now the impact of the inclusion of the Yukawa interaction $\mathcal{W} = fM^i_j q_i \bar{q}^j$ in the action of the dual theory. For simplicity of notations we will write that the gauge group is some $SU(N_c)$ and will omit tilde in all fields. The problem which we are interested in is the aspects of the behavior near the infrared fixed point.

The action takes the form

$$S = \frac{Z_q(\mu)}{4} \sum_f \int d^4x d^4\theta \left(\bar{q}^\dagger_f e^V \bar{q}_f + q^\dagger_f e^{-V} q_f \right) + \frac{Z_M(\mu)}{4} \int d^4x d^4\theta \; M^\dagger M + \frac{1}{2g^2(\mu)} \int d^4x d^2 \mathrm{Tr}\, \theta \; W^2 + \left[f_0 \int d^4x d^2\theta \; M^i_j q_i \bar{q}^j + h.c. \right] \tag{51}$$

where Z_q and Z_M are the quark and M field Z factors and f is the Yukawa coupling. The Yukawa interaction is the F term and due to the famous nonrenormalization theorem [22] f_0 does not depend on μ; the renormalized coupling is unambiguously defined as

$$f(\mu) = f_0 / (Z_q(\mu) \sqrt{Z_M(\mu)})\,. \tag{52}$$

The renormalization group equation for $f(\mu)$ then has the form

$$\frac{df^2}{d\ln\mu} = f^2 \;\; [\gamma_M(\alpha, f) + 2\gamma_q(\alpha, f)] \tag{53}$$

where the quark and meson anomalous dimensions are defined in the standard way

$$\gamma_q(\alpha, f) = -d\ln Z_q / d\ln\mu, \qquad \gamma_M(\alpha, f) = -d\ln Z_M / d\ln\mu\,. \tag{54}$$

If we believe in the conformal window we must insist that there is an IR fixed point for both couplings – gauge and Yukawa. Then one of the conditions of the IR fixed point (α_*, f_*) will be

$$\gamma_M(\alpha_*, f_*) + 2\gamma_q(\alpha_*, f_*) = 0 \tag{55}$$

(Let us parenthetically note that one can obtain the condition for the zero of β function (55) by studying the multiplet of anomalies. This was done in a recent paper by Leigh and Strassler [21] who discussed marginal operators in different $N = 1$ and $N = 2$ theories.)

One can make very general statements about the asymptotic behavior of the Z factors in the infrared region. The first one is that $Z_q \to 0$ when approaching the

conformal point. To see this is indeed the case let us start from a particular example, $f=0$. Then $Z_q(\mu)=Z_q(\alpha(\mu))$ because of the renormalizability of the theory, and

$$-\gamma_q(\alpha)=\frac{d\ln Z_q(\alpha(\mu))}{d\ln\mu}=\frac{d\ln Z_q(\alpha)}{d\alpha}\frac{d\alpha(\mu)}{d\ln\mu}=\beta(\alpha)\frac{d\ln Z_q(\alpha)}{d\alpha}\,,$$
$$Z_q(\alpha)=Z_q(0)\exp\left(-\int_{\alpha_0}^{\alpha}d\tau\frac{\gamma_q(\tau)}{\beta(\tau)}\right)\,. \tag{56}$$

For definiteness, the bare Z-factor $Z(0)$ can be set equal to unity. It is obvious that near fixed point α_*, where the β function is zero, only the singular behavior of $1/\beta(\alpha)$ is important, and the leading contribution comes from the upper limit of integration $\alpha\to\alpha_*$. Therefore, one can substitute $\gamma_f(\tau)$ by a constant, $\gamma_f(\alpha_*)$. The β function behavior near the IR fixed point is $\beta(\tau)=-\beta'(\alpha^*)(\alpha^*-\tau)$, where $\beta'(\alpha^*)$ is positive in the vicinity of the IR fixed point. After integration in (56) one gets

$$Z_q(\alpha)\sim\exp\left(-\frac{\gamma_q(\alpha_*)}{\beta'(\alpha_*)}\ln(\alpha_*-\alpha)\right)=(\alpha_*-\alpha)^{-\gamma_q(\alpha_*)/\beta'(\alpha_*)}\,. \tag{57}$$

Since $\gamma_q(\alpha_*)<0$ one immediately concludes that at

$$\alpha\to\alpha_*,\qquad Z_q(\alpha)\to Z_q(\alpha_*)=0\,. \tag{58}$$

In the general case, $f\neq 0$, the same result, $Z_q\to 0$ in the IR fixed point, can be easily obtained by examining the following renormalization group invariant combination

$$I\sim\exp\left(-\frac{2\pi}{\alpha(\mu)}\right)\mu^{(3N_c-N_f)}\left(\frac{2\pi}{\alpha(\mu)}\right)^{N_c}Z_q^{-N_f}(\mu) \tag{59}$$

which follows from the NSVZ β function (1). The dependence on the Yukawa coupling f enters only through $Z_q(\alpha,f)$. Now if we reach the IR fixed point (α_*,f_*) at infinitely large spatial scale, i.e. at zero μ, to keep I invariant we must have

$$Z_q(\alpha_*,f_*)=0 \tag{60}$$

(and this is the only possibility, because α_* is finite). The condition (55) implies then that $Z_M(\alpha_*,f_*)\to\infty$ provided that f_* is finite.

The conformal point for both coupling constant is defined now as the point of the intersection of curve (55) with the curve

$$N_f\gamma_q(\alpha_*,f_*)+(3N_c-N_f)=0\,. \tag{61}$$

In the general case we do not know how to find this intersection point which is crucial for the existence of the conformal window. The consideration summarized in Sec. 2 assumes that the intersection does exist. We can *prove* that it exists in

the weak coupling regime. In this case it is not difficult to obtain the whole phase portrait of the RG flow for two couplings – $\alpha(\mu)$ and $f^2(\mu)$ (when both of them are small). To this end the one- loop results for $\gamma_q(\alpha, f)$ and $\gamma_M(\alpha, f)$ can be used. It is clear that the one-loop contribution to $\gamma_M(\alpha, f)$ does not depend on α, because M is a singlet with respect to $SU(N_c)$. It is very easy to calculate this anomalous dimension; the result is

$$\gamma_M(\alpha, f) = \frac{f^2}{8\pi^2} N_c + o(f^2, \alpha) \tag{62}$$

It is also clear that the same formula (with the substitution $N_c \to N_f$) will describe the Yukawa coupling contribution to the anomalous dimension of the quark superfield. Combining with (43) we get

$$\gamma_q(\alpha, f) = -\frac{\alpha}{\pi}\frac{N_c^2 - 1}{2N_c} + \frac{f^2}{8\pi^2} N_f + o(f^2, \alpha)\,. \tag{63}$$

Two different signs appearing in Eq. (63) are perfectly transparent. The Yukawa contributions to anomalous dimensions must be positive. In pure Yukawa theory one can not get asymptotic freedom and the β function must be positive, which means positivity for any anomalous dimensions at zero α. On the other hand, the gauge coupling contribution must be negative. Indeed, the same Z factor renormalization is responsible for the running of mass, which is always asymptotically free in gauge theories.

At small f^2 and α the following RG equations take place

$$\begin{aligned} \frac{d\alpha}{d\ln\mu} &= -\frac{\alpha^2}{2\pi\left(1 - (N_c\alpha/2\pi)\right)}\left[3N_c - N_f + N_f\left(N_f\frac{f^2}{8\pi^2} - \frac{N_c^2-1}{2N_c}\frac{\alpha}{\pi}\right)\right]\,, \\ \frac{df^2}{d\ln\mu} &= f^2\,\left[\frac{f^2}{8\pi^2}(N_c + 2N_f) - \frac{\alpha}{\pi}\frac{N_c^2-1}{2N_c}\right]\,. \end{aligned} \tag{64}$$

The corresponding phase portrait is given on Fig. 1. It is easy to see that for $0 < 3N_c - N_f \ll (N_c,\ N_f)$, one has the infrared attractive fixed point

$$\begin{aligned} \frac{\alpha_*}{\pi} &= \frac{3N_c - N_f}{N_f}\cdot\frac{2N_c}{N_c^2-1}\cdot\frac{N_c + 2N_f}{N_c + N_f}\,, \\ \frac{f_*^2}{8\pi} &= \frac{3N_c - N_f}{N_f}\,(N_c + 2N_f)\,. \end{aligned} \tag{65}$$

Notice that the old fixed point (i.e. the one in the theory with the zero Yukawa coupling) becomes unstable.

When $3N_c - N_f$ becomes larger we must go beyond the one-loop approximation for γ's to find the IR fixed point which is defined as the intersection point for two curves (55) and (61). Although the existence of the IR fixed point is proven only near the right edge of the window it is natural to expect that qualitatively the

same picture holds in the whole window, up to $N_f = (3/2)N_c$. Logically one cannot exclude, however, that at large $3N_c - N_f$ these two curves do not intersect each other at all – the RG flow in this case will lead us towards the strong coupling region where in any case we must take into account the effects of the pole at $\alpha_p = 2\pi/N_c$ in the gauge coupling β function.

Note that there exists a domain in the $\{\alpha\,,\ f^2\}$ plane where the theory is asymptotically free with respect to both constants α and f^2. In this domain the argument of Sec. 3 is applicable literally. Outside this domain generically we find ourselves in the situation with the Landau pole for f^2, so that f^2 becomes large in the limit of vanishing distances. Even in this regime, however, there exists a domain of intermediate distances where f^2 is small (see Fig. 1), so that the inequivalence demonstrated in Sec. 3 will take place in this intermediate domain for sure.

Summarizing, we have demonstrated that Seiberg's IR fixed point in the theory without the Yukawa coupling is unstable; the theory evolves towards a new stable IR fixed point (given by (65) in the weak coupling domain). All quark Z_q factors go to zero near this fixed point. At the same time the meson Z_M factor diverges near this fixed point as $1/Z_q^2$.

Let us also mention the possibility of adding the bare mass terms like $m_0\bar{q}q$ in the theory with the conformal point. Because $Z_q \to 0$ even the infinitesimally small m_0 will lead to a large renormalized mass $m = m_0/Z_q$, and the RG flow, instead of approaching the IR conformal point, will be diverted to the strong coupling region. It is very important to take into account the effect of the vanishing Z factors when considering the mass deformations, for example, in the case of a small explicit SUSY breaking. At the same time the bare mass term for the meson field M will go to zero near the IR fixed point. If one adds both mass and interaction terms for the M field,

$$\int d^4x d^2\theta \left(m_0^2 M + \lambda_0 M^3 \right) , \tag{66}$$

the renormalized values for m and λ are

$$m(\mu) = m_0/\sqrt{Z_M(\mu)}, \quad \lambda(\mu) = \lambda_0/Z_M^{3/2}(\mu)\,. \tag{67}$$

They tend to zero at the fixed point, and at the same time the vacuum expectation value of the renormalized scalar field $M_R = M/Z_M^{1/2}$ goes to infinity

$$\langle M^2 \rangle \sim Z_M(\mu)(m_0^2/\lambda_0)\,, \tag{68}$$

as well as the quark mass due to the Yukawa coupling $f_0\ M\bar{q}q$ which will diverge as $m \sim (m_0/\lambda^{1/2})(f_0/Z_q(\mu))$, i.e. in the same way as in the case of the bare quark mass.

9 Conclusions

The infrared duality between two different $N = 1$ theories advocated in Ref. [1] seems to be a very promising direction in the studies of the nonperturbative gauge

field dynamics. We have shown that the standard 't Hooft consistency conditions in the external backgrounds, crucial in establishing the duality, receive no contributions of the higher order in α and f although the higher order corrections enter in the definition of the conserved R current. In particular, these higher order corrections are responsible for the fact that the naive (one-loop) R current defined in the UV flows to R^0 residing in the same supermultiplet with the energy-momentum tensor in IR. Due to specific holomorphy properties, the corrections in the R current cancel those appearing in the anomalous 't Hooft triangles, so that the net result of Ref. [1] remains intact. The one-loop nature of the 't Hooft consistency conditions in the supersymmetric theories with the matter fields and a conserved R current is the most important practical lesson of our analysis.

The requirement of the conformal symmetry in the infrared limit in the magnetic theory yields a constraint on the anomalous dimension of the M field. If this constraint is satisfied the magnetic theory matches the electric one in the infrared limit. It is important that the conformal symmetry must apply to all interactions, including the Yukawa interaction in the magnetic theory. It is also amusing, that only in the stable IR fixed point the matching will take place - as we saw there is an unstable IR fixed point, where the Yukawa coupling is zero - but there is no matching in this point.

Even if duality is achieved in the infrared limit, there is no way the electric and magnetic theories can coincide identically.

We established some simple facts about Z factors relevant to the conformal window and the infrared attractive fixed points. A very general argument about nullification of the quark Z factors was presented and possible consequences discussed in brief. Since the old IR fixed point turns out to be unstable it is important to include the Yukawa couplings from the very beginning. In the weak coupling regime the whole phase portrait in the two-coupling plane is established.

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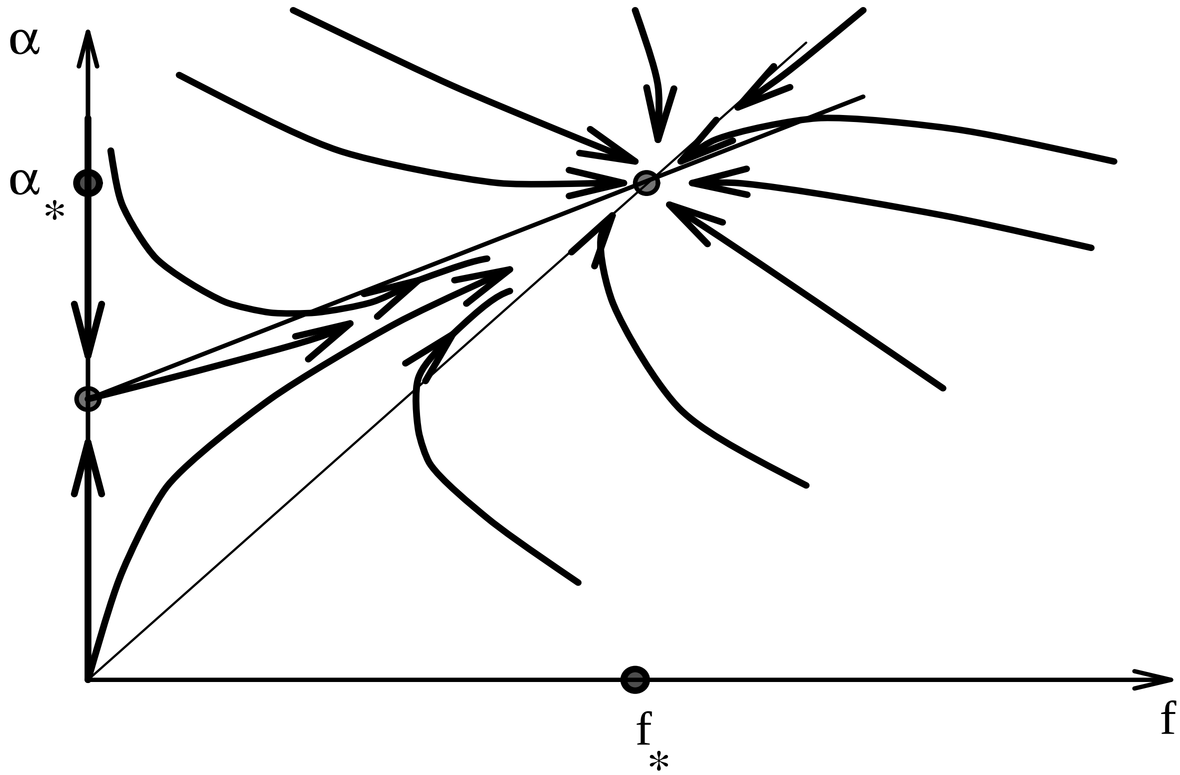

Fig.1